factors presented by the T and image guides are higher than the ones presented by the π equivalent guide [4]. Furthermore, the transversal decay of the electromagnetic field on the π guide is much lower than it is on the image or T guides. Consequently one has to use to use higher minimal curvature radii for the π guide, and thus one has to enlarge the circuit dimensions.

Reply[2] *by J. F. Miao and T. Itoh*[3]

The authors of the paper [1] thank the authors of the above comments for drawing attention to the existence of excellent works reported elsewhere. Due to limited communication skills, the authors of [1] could not detect the papers referenced in the comments. It should be noted, however, that the primary objective of [1] is to develop a directional coupler with additional design parameters after a simple theory of analysis is experimentally confirmed.

The generalized telegrapher's equation by Schelkunoff has previously been used by Ogusu in analyzing a number of dielectric waveguides [5].

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Comment on "Variational Methods For Nonstandard Eigenvalue Problems in Waveguide and Resonator Analysis"

G. J. GABRIEL

In his recent work[1], Lindell proposes variational methods for so-called nonstandard eigenvalue problems with sweeping generalities. To be sure, unorthodox formulations and solution of broad classes of problems ought to be encouraged, provided that they are consistent and that they are cogently demonstrated to offer significant improvement over existing knowledge. It is the purpose of the following comments to prove the inherent fallacy of the proposed principle and to call attention to aspects of this work that seriously deviate from established principles and usage of mathematics. These deviations offer a partial explanation for failure of the proposed method. For convenience, whenever equations are reproduced here they are numbered as the original work followed with the suffix L.

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[1]I. V. Lindell, *IEEE Trans. Microwave Theory Tech.*, vol. MTT 30, no 8, Aug. 1982.

Proof of Fallacy

According to Lindell, the nonstandard eigenvalue problem is formulated as

$$L(\lambda)f(\boldsymbol{r})=0, \qquad \boldsymbol{r}\in S \tag{1L}$$

$$B(\lambda)f(\boldsymbol{r})=0, \qquad \boldsymbol{r}\in C \tag{2L}$$

where f is a vector field defined at points of a two-dimensional plane S bounded by a closed curve C, a subset of S. The operator $L(\lambda)$ is linear and depends on a parameter λ, interpreted as the eigenvalue, and $B(\lambda)$ is another linear operator which primarily states the boundary constraint on f at points on C. To arrive at the variational principle, the rather unorthodox notion of boundary inner products is introduced resulting in the so-called generalized Green Theorem stated in (3L). It is then asserted that the linear functional

$$F(\lambda;f)=(f,L(\lambda)f)+(Cf,B(\lambda)f)_b \tag{5L}$$

is a variational principle in that when $F(\lambda;f)=0$, then the variation δF vanishes whenever the variation $\delta\lambda$ vanishes, provided that f is a solution of the system (1L) and (2L). Here the subscript b denotes integration over the boundary.

To prove the fallacy of this assertion, let us for the moment accept the notion of inner product on the boundary, the generalized Green Theorem (3L), and the ill-defined meaning of adjoint operator. Then, by pure formalities, there results

$$\delta F(\lambda;f)=2\left[(\delta f,Lf)+(C\delta f,Bf)_b\right]+\delta\lambda\left[(f,L'f)+(Cf,B'f)_b\right] \tag{6L}$$

where L' and B' are operators denoting derivatives of L and B with respect to λ. Here, Lindell has overlooked the fact that f may be a function of λ, to say nothing for the moment of the alarming presence of variations δf on the boundary! Nevertheless, he maintains that δF vanishes if $\delta\lambda$ vanishes "unless by chance the (second) bracketted term is zero. Hence, if we solve for λ the equation $F(\lambda;f)=0$ the arising functional $\lambda=J(f)$ is stationary when f is a solution of (1L) and (2L) and the stationary value of $J(f)$ is the value of the corresponding parameter, the nonstandard eigenvalue."

Aside from overall improprieties inherent in the formulation as discussed below, the sufficiently decisive question here is whether or not the second bracketted term in (6L) vanishes. Unfortunately, this term which we denote by $A(\lambda)$ does vanish always, as demonstrated next. It is presumed that L' and B' are proper derivatives of operators with respect to λ. Since f is a solution of (1L) and (2L), it also is a function of λ. It follows from (1L) and (2L) that

$$L'(\lambda)f=-L(\lambda)\frac{\partial f}{\partial\lambda} \tag{1}$$

$$B'(\lambda)f=-B(\lambda)\frac{\partial f}{\partial\lambda} \tag{2}$$

where on the righthand side L and B operate on $\partial f/\partial\lambda$. If f is

not a function of λ, as is the case in the derivation of (6L), then both $L'f$ and $B'f$ vanish identically. More appropriately, if we take into account the fact that f is a function of λ, then the consistent form of the second braketted term would be

$$A(\lambda)=\left(f, L'f+L\frac{\partial f}{\partial\lambda}\right)+\left(Cf, B'f+B\frac{\partial f}{\partial\lambda}\right)_b + \left(\frac{\partial f}{\partial\lambda}, Lf\right)+\left(C\frac{\partial f}{\partial\lambda}, Bf\right)_b. \quad (3)$$

Again, by (1) and (2), the first two terms vanish, and by (1L) and (2L) the last two terms vanish. Thus, $A(\lambda)$ is always zero. Q.E.D.

Because $A(\lambda)$ vanishes, it is clear from (6L) that the variation δF vanishes for *any* nonzero $\delta\lambda$. Consequently, the quantity $J(f)$ is not the stationary value of the so-called eigenvalues as Lindell maintains and erroneously applies to subsequent examples.

Stated rigorously, without a licentious use of boundary inner products, the proposed principle in (5L) is tantamount to the following. Given the linear operator $L(\lambda)$: $D\rightarrow D$, mapping the domain D to the range D, and suppose for simplicity that the system $[L; D]$ is self-adjoint in the accepted sense discussed below. We seek those values of $\lambda=J(\boldsymbol{u})$ which, for any vector $\boldsymbol{u}\in D$, result in the vanishing inner product

$$(\boldsymbol{u}, L(\lambda)\boldsymbol{u})=0. \quad (4)$$

Obviously, *a sufficient but not necessary* condition for this inner product to vanish is that $\boldsymbol{u}$ be a solution of $L\boldsymbol{u}=\boldsymbol{0}$. However, the roots $\lambda=J(\boldsymbol{u}_o)$ of (4) calculated for any $\boldsymbol{u}_o$ such that $L\boldsymbol{u}_o\neq\boldsymbol{0}$ also reduce the inner product to zero so that the most one can assert is that, for these particular values of λ, $\boldsymbol{u}_o$ and $L\boldsymbol{u}_o$ are orthogonal. There is nothing here that necessarily suggests a stationary character or that $J(\boldsymbol{u}_o)$ is an eigenvalue.

The ambiquity is perhaps best illustrated by examining one of the examples, which is typical of the others, offered by Lindell in support of his exposition. Stripped of the above formalism, the example in Section II-B is essentially stated as

$$\nabla\times\nabla\times\boldsymbol{f}(\boldsymbol{r})-\lambda^2\mu\epsilon\boldsymbol{f}(\boldsymbol{r})=\boldsymbol{0}, \qquad \boldsymbol{r}\in S \quad (16L)$$

$$\boldsymbol{n}\times\nabla\times\boldsymbol{f}(\boldsymbol{r})=\lambda K\boldsymbol{f}_t(\boldsymbol{r}), \qquad \boldsymbol{r}\in C \quad (17L)$$

where $\boldsymbol{n}$ is the unit inward normal on C and $\boldsymbol{f}_t=\boldsymbol{f}-\boldsymbol{nn}\cdot\boldsymbol{f}$. Directly, we form the inner product and set it equal to zero

$$(\boldsymbol{f}, L\boldsymbol{f})=\int_S \boldsymbol{f}^*\cdot(\nabla\times\nabla\times\boldsymbol{f}-\lambda^2\mu\epsilon\boldsymbol{f})\,dS=0. \quad (5)$$

However, it should be noted that although $\boldsymbol{f}$ has all three Euclidean components, they are not functions of the coordinate z. So, in effect, the operator ∇ is the usual transverse operator ∇_T given in the coordinates of S. By routine application of the vector identity

$$\nabla_T\cdot(\boldsymbol{g}\times\boldsymbol{f})=\boldsymbol{f}\cdot\nabla_T\times\boldsymbol{g}-\boldsymbol{g}\cdot\nabla_T\times\boldsymbol{f} \quad (6)$$

together with the two-dimensional divergence theorem and the boundary condition (17L), one arrives at the expression for $F(\lambda; \boldsymbol{f})$ given here as

$$F(\lambda, \boldsymbol{f})=-\lambda^2\mu\epsilon\int|\boldsymbol{f}|^2\,dS-\lambda K\int|\boldsymbol{f}_t|^2\,dC+\int|\nabla\times\boldsymbol{f}|^2\,dS=0. \quad (20L)$$

This is not the simplistic quadratic form in λ that it is intimated to be. On the one hand, if $\boldsymbol{f}$ is to be a solution of (16L), then it must necessarily be a function of λ. The integrals of $\boldsymbol{f}$ comprising the coefficients of λ^n are then also functions of λ. Consequently, the expression $F(\lambda; \boldsymbol{f})=0$ is in reality a transcendental equation which would usually have an infinite sequence of roots. On the other hand, if $\boldsymbol{f}$ is not a solution of (16L) but it is *any* integrable and once differentiable function selected at will, the resulting roots λ assure at most that $\boldsymbol{f}$ is orthogonal to $L\boldsymbol{f}$, as a solution to (5).

These considerations can be demonstrated more concretely by taking

$$\boldsymbol{f}(\boldsymbol{r})=\boldsymbol{u}_z\sin\alpha x$$

over the rectangular region $x\in[0, a]$, $y\in[0, b]$. By forcing $\boldsymbol{f}$ to be a solution of (16L) we must have $\alpha=\lambda\sqrt{\mu\epsilon}$. In that case, after calculation of the indicated integrals in (20L), it would be easy to see that the result is a transcendental equation whose roots are, for $n=0, \pm1, \pm2, \cdots$,

$$\lambda_n=\frac{c}{a}[2n\pi+\arctan(1/Kc)]$$

where $c=1/\sqrt{\mu\epsilon}$. Alternately, an arbitrary α would lead to the quadratic roots λ which assure the vanishing inner product (5) but not a solution of (16L). What, then, is the distinction between the two sets of roots? The proposed theory does not provide unambiguous answers, if any.

Deviations from Convention

Failure of the proposed variational principle is not surprising in light of the numerous departures from established mathematical principles and usage without convincing arguments. We raise two objections to elicit the sort of inconsistencies that contribute to the failure, in one way or another.

The first objection concerns treatment of the boundary terms. In boundary value problems, generally the boundary constraints serve to specify the domain of an operator. Thus in the original statement of the problem, (1L) and (2L), the domain of the functions $f(\boldsymbol{r})$ is the set of points comprising the two-dimensional plane S bounded by the curve C. In contradistinction, the domain D of the operator $L(\lambda)$ is the set of functions $f(\boldsymbol{r})$ satisfying the boundary constraint (2L), viz.

$$D=\{f(\boldsymbol{r}): B(\lambda)f(\boldsymbol{r}_o)=0, \boldsymbol{r}_o\in C\}.$$

Proper treatment of boundary constraints becomes more critical if an adjoint operator and its domain are to have the customary meanings [2], [3]. Let L: $U\rightarrow V$ be a linear operator, and L^+: $U^+\rightarrow V^+$ be another linear operator, as yet unspecified. For any pair of vectors $\boldsymbol{u}\in U$ and $\boldsymbol{u}^+\in U^+$, one forms the inner product $(\boldsymbol{u}^+, L\boldsymbol{u})$. By a process of integration by parts there results

$$(\boldsymbol{u}^+, L\boldsymbol{u})-(L^+\boldsymbol{u}^+, \boldsymbol{u})=C(\boldsymbol{u}^+, \boldsymbol{u})$$

where $C(\boldsymbol{u}^+, \boldsymbol{u})$ is called the *conjunct* and it is the integrated terms evaluated at the boundary. The boundary constraint on $\boldsymbol{u}^+$, which serves to define the domain U^+, is so selected that the conjunct $C(\boldsymbol{u}^+, \boldsymbol{u})$ *vanishes* for every $\boldsymbol{u}$ and $\boldsymbol{u}^+$. In that case, $[L^+; U^+]$ is the adjoint of the system $[L; U]$. Unless the conjunct vanishes, the adjoint system is not fully defined or, at best, it is an improper adjoint. When $L=L^+$ and $U=U^+$, the system is self-adjoint.

What Lindell calls boundary inner product, without justification or rigor, is what is commonly called the conjunct. Indeed, if this inner product is consistent, its significance should also hold in the simplest case of the one-dimensional scalar Green theorem. For $u(x)$ and $v(x)$ defined over the interval $a\leqslant x\leqslant b$, we have

$$\int_a^b(uv''-u''v)\,dx=W_b(u, v)-W_a(u, v)=C(u, v)$$

where W_b and W_a are the Wronskians evaluated at b and a, the end points. One would be hard pressed, by any measure of

consistency, to interpret these boundary terms as inner products, let alone interpretting *values* of derivatives at a point as an operator within the inner product. The objectionable nature of this interpretation of operators on the boundary is further illustrated if one considers the usual three-dimensional Green Theorem over a sphere

$$\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) dv = \int_\Sigma (\phi\nabla\psi - \psi\nabla\phi)\cdot \boldsymbol{n}\, dS = C(\phi,\psi).$$

Here, also, one sees that the derivatives at the boundary are with respect to the radial coordinate evaluated at a fixed radius, and not operators in the angle coordinates of integration. Lindell's proposition is neither consistent nor sufficiently compelling to warrant departure from the customary viewpoint.

The second objection centers on the very notion of a nonstandard eigenvalue problem which is found tenuous if not meaningless in a variational context. In essence, the calculus of variations seeks to identify those functions (vectors) that extremize a certain functional which is usually in the form of a definite integral [4]. The admissible functions form a class restricted by the common boundary constraint shared by all the functions at the boundary of the region of integration. Consequently, as a matter of course, the variations, δf, must vanish at the boundary in any proper variational development. The very inclusion of a boundary term in Lindell's expression is sufficient to question its veracity. Moreover, a variational expression, as such, is not an approximation method, though it is sometimes convenient to employ it that way. Rather, it is a principle which states in a nutshell an essential property of a system from which the governing equations of the system are obtained. Thus for example, minimization of the arc length between two points in a given metric leads to equations for geodesics, minimization of the action integral over a fixed time interval leads to Newton–Lagrange equations of motion, minimization of electrostatic energy leads to the Laplace Equation, and so on.

The eigenvalue problem as it is commonly understood is not different. However, not every parameter of a system qualifies as an eigenvalue, and not every solution of a problem is an eigenvector, as asserted in [1]. Variational aspect of an eigenvalue problem may be summarized thus. Given a linear self-adjoint positive-definite operator L: $U \to V$. In order for an inner product $(\boldsymbol{u}, \boldsymbol{v})$ to be meaningful for every $\boldsymbol{u} \in U$ and $\boldsymbol{v} \in V$, it is necessary that both U and V be subsets of a common set. The object is to identify those vectors $\boldsymbol{u}$ whose projection on $\boldsymbol{v} = L\boldsymbol{u}$ is a maximum subject to normalization with respect to $\|\boldsymbol{u}\|$. In other words, the vector $\boldsymbol{u}$ must be so chosen as to maximize the non-linear functional

$$F[\boldsymbol{u}] = \frac{(\boldsymbol{u}, \boldsymbol{v})}{(\boldsymbol{u}, \boldsymbol{u})} = \frac{(\boldsymbol{u}, L\boldsymbol{u})}{(\boldsymbol{u}, \boldsymbol{u})}. \tag{8}$$

Analytical proofs of the extremum property of this functional are quite well known [2]. Here, a simple geometric analogy is sufficient for us to conclude that the projection is maximized when $\boldsymbol{u}$ and $\boldsymbol{v}$ are colinear. That is, if there exists a number α, called an *eigenvalue*, and a vector $\boldsymbol{u}_\alpha$, called an *eigenvector*, such that

$$L\boldsymbol{u}_\alpha = \alpha \boldsymbol{u}_\alpha \tag{9}$$

then $F[\boldsymbol{u}_\alpha]$ is a maximum and *its value is* $F[\boldsymbol{u}_\alpha] = \alpha$. This is the commonly understood meaning of an eigenvalue problem.

The choice of $\|\boldsymbol{u}\|$ for normalization is not critical. For, if there is another operator M: $U \to W$, and $\boldsymbol{w} = M\boldsymbol{u}$, one could maximize the projection of $\boldsymbol{u}$ on $L\boldsymbol{u}$ subject to normalization with respect to the projection of $\boldsymbol{u}$ on $\boldsymbol{w}$, viz.

$$F[\boldsymbol{u}] = \frac{(\boldsymbol{u}, L\boldsymbol{u})}{(\boldsymbol{u}, M\boldsymbol{u})}. \tag{10}$$

This functional is maximum when

$$L\boldsymbol{u}_\beta = \beta M\boldsymbol{u}_\beta. \tag{11}$$

Eigenvalue problems encountered in waveguide theory belong in this class as first shown in earlier works [5], [6]. Extension of the projection viewpoint to operators that are not self-adjoint is straightforward.

The critical point to emphasize here is that the eigenvalue is a particular (maximum) value of the functional $F[\boldsymbol{u}]$ and *not* the value of some parameter that appears in the functional through the dependence of the operator on that parameter. In the proposed nonstandard eigenvalue problem [1], the values of a functional generated from projection are being confused with values of the parameter λ which, in a generic form $L(\lambda)$, properly belongs to the inherent structure of the operator. Allowing the parameter λ to vary essentially changes the operator and, therefore, the problem because for $\lambda_1 \neq \lambda_2$, the operators $L(\lambda_1)$ and $L(\lambda_2)$ are not the same. Accordingly, when properly formulated as in (9), the eigenvalues of $L(\lambda)$ would be functions $\alpha(\lambda)$. Moreover, the functional in (8) is extremized relative to choices of vectors from the space on which it is defined, *but not relative to variations in* λ. Simply stated, λ as employed in [1], in both the abstract formalism and examples, cannot be an eigenvalue in the accepted sense. What are really being calculated in these specific examples are those values $\lambda = \lambda_n$ for which $L(\lambda_n)$ has the zero eigenvalue, $\alpha_0(\lambda_n) = 0$, corresponding to a vanishing wavenumber in waveguides. As further evidence, despite the fact that $\nabla \times \nabla \times$ is a self-adjoint operator on the indicated domain in (16L) and (17L) discussed above, with real K and λ, solutions f_m and f_n corresponding to λ_m and λ_n, if such exist, are not orthogonal as one would expect of proper eigenvectors. This can be readily demonstrated.

Conclusion

The foregoing may seem to be a narrow viewpoint of what an eigenvalue problem is. However, this is necessary because if the conditions of a particular theory or method are not met in a given situation, then the assertions—e.g. orthogonality—derived from that theory are not applicable with any measure of confidence or legitimacy. An inconsistent formalism that fortuitously agrees with some measurement or some calculations proves nothing. In contrast, a consistent theory that disagrees with measurements does have a value, in that it logically points to inadequacy of the hypothesis made about the physical system. In this sense, the numerical calculations offered in [1] in support of the theory are of dubious merit as supportive evidence, since the underlying theory to begin with is proven inconsistent.

Reply[2] by I. V. Lindell[3]

The above comments on my paper given by Dr. Gabriel were received with interest, in fact, some reaction from the mathematicians was expected, because the concepts given in [1] cannot be

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considered final and there seems much honing to be done by mathematicians before the theory can satisfy everybody. The present comments, however, did not quite meet my expectations. In fact, I was puzzled to find that in spite of the evidence given in [1], the reaction was so negative. The critique is not only towards myself, but also towards the system of refereeing in this TRANSACTIONS because [1] and two further papers applying the theory of [1] have penetrated the refereeing sieve.

A close consideration of the comments by Dr. Gabriel, however, reveals an ignorance of the main logic presented in [1]. The central argument considers the validity of my expression denoted by (6L) in Dr. Gabriel's comments. Instead of proving fallacies in these comments (which would be of entertaining value to readers with interest in debates), I would prefer to restate my arguments in a clearer manner (which is more useful for those really interested in the method presented in [1]).

To be as concise as possible, let us drop the boundary condition operator and suppose that the functions considered satisfy these conditions. So, the (nonstandard) eigenvalue equation is simply of the form $L(\lambda)f = 0$. Since we do not consider radiation problems, the eigenvalues form a discrete set λ_n. The eigenfunction f is, of course, a function of the eigenvalue, but since $f = 0$ for those values of λ which do not belong to the spectrum of eigenvalues, the functional relationship is not a continuous one, in fact, there is no derivative $\partial f/\partial\lambda$ at the eigenvalues, at least in the traditional sense.

The central point is whether we can obtain stationary functionals from the equation $(f, L(\lambda)f) = 0$ or not. To make no room for misunderstanding, let us consider an equation of the form $(g, L(\beta)g) = 0$, where β is a parameter and g a function satisfying the correct boundary conditions. It can be well assumed that this equation possesses roots β, which depend on the function g continuously. It also possesses the (nonstandard) eigenvalues λn as roots when g equals f, the corresponding eigenfunctions. From the continuity we deduce that when g departs from f just a little, say, $g = f + \delta f$, the corresponding root also departs from λ just a little: $\beta = \lambda + \delta\lambda$. The equation for $\delta\lambda$ is obtained from the first-order equation $2(\delta f, L(\lambda)f) + \delta\lambda(f, L'(\lambda)f) = 0$. Because the first term vanishes for solutions of the eigenvalue equation, so must the second. Hence, either we have $(f, L'(\lambda)f) = 0$ or $\delta\lambda = 0$. As we see from the above, the dependence of f on λ does not have any role, whence there is no reason to assume that the bracketed term vanishes, in general, which leads to the condition $\delta\lambda = 0$, or the roots of the equation $(g, L(\beta)g) = 0$ approximate $\beta \approx \lambda$ up to the second order if g approximates f to the first order.

The present theory is best demonstrated by examples, which are not mere calculations, as Dr. Gabriel calls them, but methods of obtaining stationary functionals, whose stationarity can be checked at once. In addition to the examples mentioned in [1], there are more complicated problems in [7] and [8] which rely on the present theory. The example given in Dr. Gabriel's comments appears strange to me. The function $uz\sin(\alpha x)$ does not satisfy the correct boundary conditions. Choosing a proper function will result in the correct eigenvalue.

The second objection by Dr. Gabriel concerns the usage of mathematics, which he finds different in [1] from that in books on mathematics. To this I must admit, although boundary integrals have been present in stationary functionals even before [9]. We could take the notion of boundary inner product as a convenient way of writing down lengthy expressions. It is good that there are scientists worried about the virginity of the Queen of Sciences, but as we know from history, many concepts like negative numbers or delta functions, for example, were first received with dismay by mathematical purists. In fact, there exist theories on evolution of science which urge scientists to fight existing scientific structures for progress, [10], [11]. I really hope that there will be mathematicians to explain with sufficient rigor why the theory presented in [1] works.

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